Solvability of Equations in Elementary Functions

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Introduction

The most general problem of solvability can be formulated as follows: we have a set of "known quantities" K and a set of "allowed" operations $\mathcal{F}$. The question is, can a given element be obtained from K by repeatedly applying operations from $\mathcal{F}$? The main approach to these questions is Galois theory (algebraic, differential and topological), which connects the problem of solvability to the properties of an associated group and a method developed by Liouville, where the idea is that either the solution is "simple" or doesn't exist [1].

Historical background

In 1770 Lagrange unified various methods for solving polynomial equations in radicals [2]. The method relied on Lagrange resolvents, which allowed to reduce an equation to one of a lower degree. This method, however, does not work for higher degrees, as the new equation has a bigger degree than the original polynomial. Lagrange's work inspired Ruffini, who made 6 publications in an attempt to prove insolvability of equations of degree higher than 4, the first publication made in 1799 and the last in 1813 [3]. The first accepted proof is due to Abel [4], published in 1824. Later in that decade, Galois developed a general theory which could determine whether a given equation is solvable by radicals [5]. Galois method connects the solvability of an equation with the solvability of its Galois group. In 1833 Liouville extended the works of Abel in 1823 about expression of integrals in elementary functions [6] (Abel's work "a general representation of the possibility to integrate all differential formulas" was not published). Around 1883-1904 Picard and Vessiot associated a Galois group with differential equation, to describe when it can be solved in quadratures and this theory was further refined by Kolchin around 1950 [7]. In 1963 V.I. Arnold discovered a topological proof of the Abel-Ruffini theorem [8], which was the starting point of topological Galois theory, to which this work is devoted.

Notions from topological Galois theory

- 'Known quantities' $K = \mathbb{C}(a)$, 'allowed' operations $\mathcal{F} = \{\text{algebraic operations and elementary functions}\}$
- **Def.** *Monodromy group is the group acting on the roots of the equation when the parameter a makes a loop.*
- Monodromy group acting on an elementary function is cyclic.
- When the equation is solvable the monodromy group must be solvable.

My contribution

$\tan x - x = a$

- Calculated singular points
- Proved a lemma about the location of roots of $\tan(x) - x = 0$
- Calculated the action of the group for paths around singular points
- Proved that the monodromy group contained the alternating group

$x^x = a$

- Transforms to equation $x + e^x = a$
- Singular points were calculated
- Action of paths around these point was calculated
- Using transitivity it was shown that the group of monodromy is the symmetric group, hence unsolvable

Main result 1

The equation $\tan x - x = a$ is not solvable in elementary functions [9]

Main result 2

The equation $x^x = a$ is not solvable in elementary functions

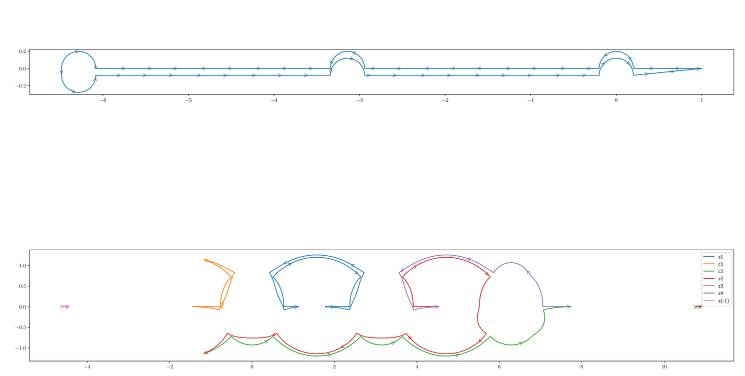

Figure 1: $\tan x - x = a$ monodromy

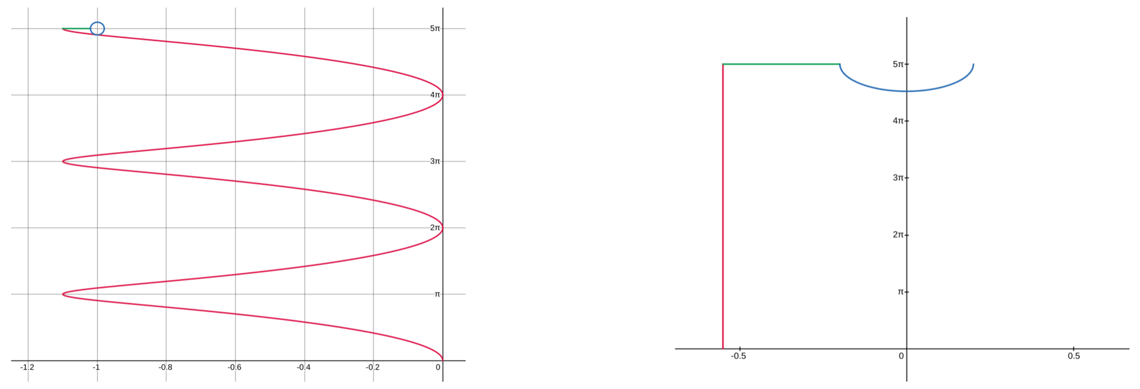

Figure 2: $x^x = a$ monodromy

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